Domain growth on self-similar structures

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The behavior of the spherical Ginzburg-Landau model on a class of nontranslationally invariant, fractal lattices is investigated in the cases of conserved and nonconserved Langevin dynamics. Interestingly, the static and dynamic properties can be expressed by means of three exponents characterizing these structures: the embedding dimensions d, the random walk exponent d_w, and the spectral dimension d_s. An order-disorder transition occurs if $d_s > 2$. Explicit solutions show that the domain size evolves with time as $R(t) \sim t^{1/d_w}$ in the nonconserved case and as $R(t) \sim t^{1/2d_w}$ in the conserved case, whereas the height of the peak of the structure factor increases in time as $t^{d_s/2}$ in the first case and as $t^{d_s/4}$ in the second while the system orders. Finally we derive the scaling function for the nonconserved dynamics and the multiscaling function for the conserved dynamics. [S1063-651X(96)03910-4]

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A system, described by an order parameter, following a quench from a high temperature disordered phase to a low temperature ordered phase undergoes a coarsening process during which the domains of the different phases compete and grow in magnitude. During the late stage of the ordering one observes patterns of characteristic length scale $R(t)$, increasing in time in a power like fashion $R(t) \sim t^{1/z}$ and the correlation functions obey dynamical scaling [1]. The exponent z has a rather universal value and in translationally invariant systems $z=2$ for a nonconserved order parameter (NCOP) evolutions, $z=3$ for scalar conserved order parameter (COP) and $z=4$ for vector COP. On the contrary, the phase separation kinetics in nontranslationally-invariant lattices remains almost unexplored, in spite of its interest. A noticeable class of models not endowed with homogeneity is the one defined on deterministic fractal supports, where translational symmetry is absent, but another property, the self-similarity is at work. Interestingly, whereas regular lattices are characterized by a single dimension d, on fractal structures at least three different dimensions are required. These are the embedding dimension d, the fractal dimension d_f of the lattice, and the spectral dimension d_s, which describes the low frequency behavior of the density of vibrational modes; on standard lattices one finds $d_f = d_s$ to coincide with the Euclidean dimension d, whereas $d_w = 2$.

In the present paper we derive the exact solution of a Ginzburg-Landau (GL) model on a fractal lattice, which we believe to be of general interest because it is representative of systems with noninteger spectral dimension, such as structurally disordered systems, percolation clusters, etc. The peculiarity of the lattice manifests itself in various ways and leads to nonrational growth exponents different from those known in the Euclidean case. Our results, besides providing an explicit realization of domain growth on fractal networks, may shed some light on relaxation phenomena occurring on complex structures.

We focus on the static and dynamical properties of a spherical version of GL model [2] on Sierpinski gaskets (SG), which can be constructed recursively [3–6] for arbitrary embedding dimension d, and have fractal dimension $d_f = \ln(d+1)/\ln 2$. The model, which is equivalent to the vector $O(\mathcal{N})$ model in the large $\mathcal{N}$ limit, can be solved as we show below and has the merit of providing useful insights on the general aspects of the physics of growth processes on nonperiodic structures.

Let us assume the order parameter ϕ_i, to be defined at each site i of SG formed by $N = (d+1)^n$ hypertetrahedra, where n represents the level of construction of the lattice.

Accordingly to the GL approach we consider the evolution of ϕ_i after a quench at temperature T_f to be governed by the equation

$$\frac{\partial \phi_i(t)}{\partial t} = -M_{il}\frac{\delta H[\{\phi_i\}]}{\delta \phi_l} + \eta_i(t) = -M_{il}\left[\Delta_{lj}\phi_j(t) + r\phi_l(t)\right.$$
$$\left. + \frac{g}{N}\sum_{k=1}^{N}\phi_k^2(t)\phi_l(t)\right] + \eta_i(t), \qquad (1)$$

where summation over repeated latin indices is assumed. In the present paper we shall consider the relevant case $r < 0$ and $g > 0$, because it accounts for phase separation.

The matrix $\boldsymbol{\Delta}$ is the discrete version of the Laplacian operator defined on the SG of embedding dimension d, while $\boldsymbol{M}$ is a kinetic operator taking the form $\Gamma\delta_{il}$ for NCOP and $-\Gamma\Delta_{il}$ for COP, where Γ is a kinetic coefficient. Through the diffusive coupling $\boldsymbol{\Delta}$ each cell is coupled on the SG to its $(d+1)$ nearest neighbors with the exception of the $(d+1)$ end vertices of the whole structure which are coupled only to d neighbors.

The noise $\eta_i(t)$ is assumed to be Gaussian with zero average and variance satisfying the fluctuation-dissipation relation $\langle \eta_i(t)\eta_j(t')\rangle = 2M_{il}T_f\delta_{lj}\delta(t-t')$, where T_f is the tem-

perature of the final state. Due to self-similarity, the spectral properties of the operator Δ can be obtained by employing a recursive algorithm [3,4]. The eigenvalue spectrum of Δ at the level of construction $(n+1)$ contains the one at level n plus new eigenvalues generated through the map

$$\lambda_n = (d+3-\lambda_{n+1})\lambda_{n+1}, \tag{2}$$

where $\lambda_n \in [0, d+3]$. After reordering the eigenvalues in ascending order and renaming them ϵ_α we employ the eigenvectors associated with the linear problem $-\Delta_{ij}u_j^\alpha = \epsilon_\alpha u_i^\alpha$ and expand the field as a linear superposition of modes $\phi_i(t) = \Sigma_{\alpha=0}^{N-1} \widetilde{\phi}_\alpha(t) u_i^\alpha$. Within the large N limit [7] we obtain the following equation for the equal-time correlation $C(\epsilon_\alpha, t) = \langle |\widetilde{\phi}_\alpha^2(t)| \rangle$:

$$\frac{d}{dt} C(\epsilon_\alpha, t) = -2\epsilon_\alpha^p \Gamma[\epsilon_\alpha + r + gS(t)]C(\epsilon_\alpha, t) + 2\Gamma\epsilon_\alpha^p T_f, \tag{3}$$

where $p=0$ for NCOP and $p=1$ for COP, respectively and we have defined

$$S(t) \equiv \frac{1}{N}\sum_{\alpha=0}^{N-1} C(\epsilon_\alpha, t) = \frac{1}{N}\sum_{i=1}^{N} \langle \phi_i^2(t) \rangle. \tag{4}$$

By the symbol $\langle . \rangle$ we denote averages over initial conditions and thermal histories.

The static equilibrium properties are found from Eq. (3) by determining self-consistently

$$\lim_{t\to\infty} S(t) = S_\infty = \frac{T_f}{N}\sum_{\alpha=0}^{N-1} \frac{1}{\epsilon_\alpha + r + gS_\infty}. \tag{5}$$

Within the spherical model the vanishing of the quantity $r + gS_\infty$ as $N \to \infty$ signals the appearance of the low temperature ordered phase below a critical temperature $T_c = -r/[gB(0)]$, where $B(0) = (1/N)\Sigma_{\alpha=1}^{N-1}\epsilon_\alpha^{-1}$ [2]. In turn, the existence of a nonvanishing critical temperature implies the finiteness of $B(0)$ as $N \to \infty$.

On the SG we have obtained the behavior of $B(0)$ in two different ways: first we observe that the sum picks out its largest contribution from the smallest elements of the spectrum, which can be approximated for n sufficiently large by

$$\epsilon_\alpha \sim E_0 \frac{1}{(d+3)^{(n-\alpha)}} = E_0 \left(\frac{2^\alpha}{2^n}\right)^{d_w}, \tag{6}$$

where E_0 is an uninteresting constant. The last equality in Eq. (6) defines the exponent $d_w = \ln(d+3)/\ln 2$, which coincides with the random walk fractal dimension on the SG [3]. Since the degeneracy of ϵ_α is proportional to $(d+1)^\alpha = 2^{\alpha d_f}$ one sees that $B(0)$ diverges as $L^{(d_w-d_f)}$, where $L = N^{1/d_f}$ is the linear size of the system in lattice units, since $d_s = 2d_f/d_w < 2$ for all d. Such a scaling is confirmed by an exact numerical calculation, see Fig. 1. Finally we have derived $B(0)$ by means of an approximation for the density of low-frequency states of the form $\rho(\epsilon) \sim \epsilon^{d_s/2-1}$. After taking the thermodynamic limit the sum over discrete eigenvalues was converted into the integral $B(0) = \int_{\epsilon_{min}} d\epsilon\, \rho(\epsilon)/\epsilon$, where for large systems the smallest
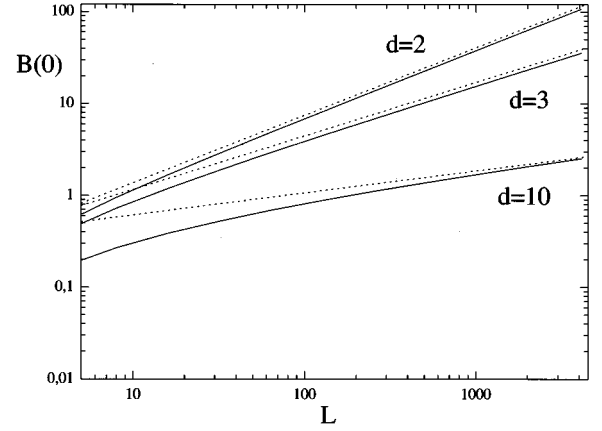

FIG. 1. Log-log plot of the sum $B(0)$ as a function of the linear size L of the system for different values of the embedding dimension d: $(d=2, d=3, d=10)$. The dashed lines represent the curves $L^{d_w-d_f}$ for each case.

positive eigenvalue $\epsilon_{min} \sim E_0 L^{-d_w}$ leading to the result $B(0) \sim \epsilon_{min}^{d_s/2-1} \sim L^{d_w-d_f}$ for $d_s < 2$. The argument shows that even a smoothed expression for $\rho(\epsilon)$ leads to the correct answer.

We must emphasize that, within the spherical model, T_c is lowered down to zero and the ordered phase shrinks along the line $T_f = 0$, for all SG of arbitrary embedding d; the existence of a finite temperature phase transition requires a spectral dimension larger than two in agreement with Refs. [5,8]. This result is a consequence of a generalized Mermin theorem [9], which states that a continuous symmetry cannot be spontaneously broken in $d_s \leq 2$.

Although the dynamical properties in the NCOP case can be determined exactly for arbitrary time t, we shall consider only the late scaling regime. Since Eq. (3) is formally linear, it can be integrated yielding for $T_f = 0$

$$C(\epsilon_\alpha, t) = C(\epsilon_\alpha, 0)\exp\{-2\Gamma[\epsilon_\alpha t + Q(t)]\}. \tag{7}$$

The auxiliary function $Q(t) = \int_0^t dt'[r + gS(t')]$, required in order to derive the scaling behavior of $C(\epsilon_\alpha, t)$ is determined by

$$\frac{dQ(t)}{dt} = r + gJ_d(t)\exp-[2\Gamma Q(t)], \tag{8}$$

where $J_d(t)$, for uncorrelated initial conditions $C(\epsilon_\alpha, 0) = C_0$, is given by

$$J_d(t) = \frac{1}{N}\sum_\alpha C_0\exp[-2\Gamma\epsilon_\alpha t] \sim C_0\int d\epsilon\, \rho(\epsilon)\exp-[2\Gamma\epsilon t]$$

$$\sim C_0 K_d t^{-d_s/2}, \tag{9}$$

where K_d is a dimensionality dependent constant. To proceed analytically we have employed the scaling ansatz for $\rho(\epsilon)$. Asymptotically we find that the correlation function assumes the form

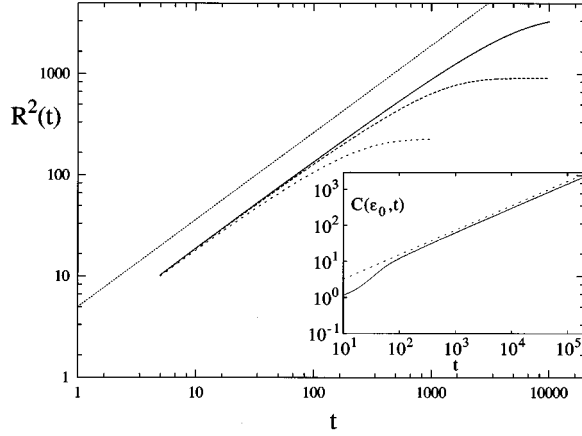$$C(\epsilon_\alpha, t) \sim C_0 t^{d_s/2}\exp-[2\Gamma\epsilon_\alpha t]. \tag{10}$$

FIG. 2. Domain size $R^2(t)$ vs t for $n=6$ (dots), $n=7$ (dashed), $n=8$ (solid), and t^{2/d_w} (short dashed line). In the inset is shown the peak of $C(\epsilon_0,t)$ vs t in the NCOP, $d=2$ case and the dashed line represents $t^{d_s/2}$.

Notice that, since the peak of Eq.(10) coincides with the zero component of the structure factor, it grows in time as $t^{d_s/2}$. In virtue of Eq. (6) we can rewrite Eq.(10) in scaling form

$$C(\epsilon_\alpha,t) \sim C_0 t^{d_s/2} \exp-[2\Gamma E_0 q^{d_w} t], \qquad (11)$$

where we have introduced the quantity $q = (2^\alpha/2^n) = (m/L)$ in order to stress the striking analogy with the standard lattice case where a similar formula holds after replacing $d_s \to d$ and $d_w \to 2$. If we insist in the parallel with the standard case we identify the prefactor $t^{d_s/2}$ with R^{d_f} and deduce $R(t) \sim t^{1/d_w}$, which represents the NCOP evolution law on fractal supports. This slowing down of the growth is caused by the delay of the diffusing particles due to the fractal structure. Let us notice that q is inversely proportional to the localization length of a state [10]. This hypothesis is corroborated by calculating numerically the quantity

$$R^2(t) = \frac{\Sigma_{ij}(i-j)^2 \langle \phi_i(t)\phi_j(t)\rangle}{\Sigma_{ij}\langle \phi_i(t)\phi_j(t)\rangle}, \qquad (12)$$

whose behavior, displayed in Fig. 2, shows the predicted scaling. To summarize, the NCOP agrees with the ordinary scaling hypothesis that the typical domain size $R(t)$ is the only relevant length during the growth and the structure function has the form $C(\epsilon,t) \sim R(t)^{d_f} F[\epsilon^{1/d_w} R(t)]$, with $F(x)$ a universal time independent shape function.

Let us turn, now, on COP dynamics by calculating the structure factor

$$C(\epsilon_\alpha,t) = C_0 e^{2\Gamma\epsilon_M^2 t} e^{-2\Gamma(\epsilon_\alpha - \epsilon_M)^2 t}, \qquad (13)$$

where $\epsilon_M = -Q(t)/2t$ represents the position of the maximum of $C(\epsilon_\alpha,t)$ and changes with time. Employing a saddle point estimate of the integrals we obtain from Eq. (4) the following approximation for the structure function:

$$C(\epsilon_\alpha,t) \sim C_0 [t^{d_s/4}(\ln t)^{(2-d_s)/4}]^{1-(1-x)^2}, \qquad (14)$$

where $x = \epsilon_\alpha/\epsilon_M$. The peak ϵ_M evolves in time as
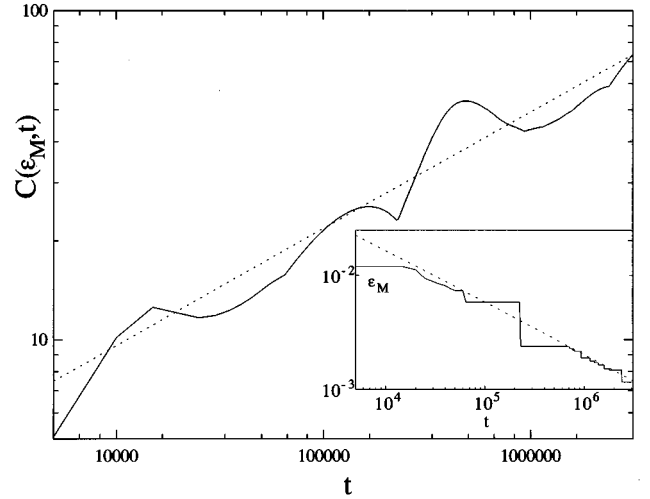

FIG. 3. Peak of $C(\epsilon_M,t)$ vs t in the COP, $d=2$ case and $t^{d_s/4}[\ln(t)]^{(2-d_s)/4}$ (dashed line) [see Eq. (14)]. In the inset we show $\epsilon_M(t)$ vs t.

$$\epsilon_M(t) = \left[\frac{d_s}{8\Gamma} \frac{\ln t + \dfrac{2-d_s}{d_s}\ln(\ln t)}{t} \right]^{1/2}. \qquad (15)$$

Since Eq. (15) shows that each mode evolves with its own exponent, the COP dynamics is characterized by multiscaling. More interestingly we observe from Fig. 3 that the height of the peak grows with the exponent $d_s/4$ on the average, but displays large deviations from a pure power law reflecting the existence of singularities in the density of states at all energy scales, a feature not observed on regular lattices.

All the results we have discussed are in total agreement with a renormalization group [2] analysis which we have carried out and will be presented elsewhere.

In conclusion, we obtain growth laws characterized by a set of noninteger exponents for an exactly solvable model with vector order parameter. For the NCOP we confirm the dynamic scaling hypothesis and derive explicitly the scaling form for the structure function [see Eq.(10)], whereas for the COP we generalize the multiscaling concept. We believe that our findings go beyond the present study and may represent a starting point to understanding phase separation occurring in complex structures, such as porous media, percolation clusters etc. Regarding the scalar version of the GL model we trust that the growth exponent z should remain d_w for NCOP, whereas for COP it would be of interest to investigate its value on self-similar lattices. On the other hand we do not expect to observe in real systems properties related to the deviation of the density of states from a smooth behavior, because the latters are characterized by a less singular density of states than deterministic fractals whose spectrum contains a high degree of correlation which manifests itself through strong correlations between different scales. On quasicrystal lattices, instead, one should observe oscillations in the peak height due to the singular spectrum, but $d_w = 2$ and $d_s = d$.

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